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On diffusive instabilities of a rapidly rotating electrically conducting layer of compressible fluid of varying depth

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Compressibility is added to Busse's (1976) study of convection in a rotating, electrically conducting layer of fluid, of varying depth, which is permeated by an azimuthal magnetic field that is orthogonal to both the rotation vector $\mathbf{\Omega}$ and the gravitational acceleration $\mathbf{g}$. On the basis of a linear theory, we investigate the kinds of infinitesimal wave motion which the fluid can support when the dominant balance of forces is between pressure gradient and Coriolis force, so that the Proudman–Taylor theorem holds. Unlike Busse, however, we assume that the fluid is statically stable in the sense that the temperature gradient is sub-adiabatic.

In the absence of diffusion, these waves are dynamically neutral and take the form of Rossby waves modified by compressibility and magnetic field. The waves are examined in two limits, the adiabatic and the isothermal, and we define two distinct frequencies at which a pure Rossby wave can oscillate. When diffusion is restored, the disparity between these frequencies makes the fluid susceptible to overstability. We prove that all such amplifying waves must propagate eastward, i.e. in the direction of $\mathbf{g} \wedge \mathbf{\Omega}$, irrespective of the sign of the depth gradient or the magnitudes of the diffusivities. The theorem does not apply if the fluid is incompressible or convectively unstable at the outset, and therefore does not contradict Busse's result that the direction of azimuthal propagation can be altered by the diffusivities. Nevertheless, we suggest reasons why Busse's method of regarding the imaginary part of the marginal stability equation as a dispersion relation is not in general a reliable one.

We examine the instabilities by subjecting the neutral waves to a weakly diffusive perturbation. We discover, in particular, a new kind of magnetic instability which is crucially dependent upon both compressibility and the depth gradient. In agreement with the general result described above, the instability takes the form of a slow, eastward propagating, amplifying wave.

The principal source for all instabilities is elastic energy, which cannot be tapped in a Boussinesq fluid, since the work done by compression is neglected under that approximation.

1. Introduction

Instability by 'magnetic buoyancy' (Parker 1955, 1977) is perhaps the most widely known example of an instability for which a magnetic field is crucial but the available potential energy is primarily not magnetic. In this particular case the non-magnetic energy is gravitational, but the *elastic* energy of a compressible fluid can also be released by a magnetic field (Acheson & Gibbons 1978). In a rotating fluid, such

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magnetic instabilities have appeared most readily in the form of amplifying waves which, relative to the rotating fluid, propagate with a frequency that is very much smaller than a typical Alfvén frequency (Roberts & Stewartson 1977; Acheson 1978*a*; Acheson & Gibbons 1978; Moffatt 1978). These 'slow waves' are typical of circumstances in which the momentum equation exhibits a 'magnetostrophic' balance between Lorentz and Coriolis forces and the dynamical (including gravitational effects) pressure gradient (see, for example, Acheson & Hide 1973). The instabilities are then intrinsically three-dimensional.

The last observation has prompted the present study: we ask whether such instabilities can take place in an almost *two*-dimensional manner. We therefore restrict our attention to a rapidly rotating [in the sense of equation (1.2)] fluid for which the dominant term in the momentum equation exhibits a balance between the Coriolis force and the (actual) pressure gradient, so that the effect of gravity and of the Lorentz force is in some sense small. Then (see, for example, Acheson 1978*b*) *steady* fluid motions are two-dimensional in filaments parallel to the rotation vector $\mathbf{\Omega}$, a result which is known as the Proudman–Taylor theorem. We shall call such a balance of forces geostrophic, even though we have solar applications of our theory primarily in mind. In this paper we shall consider 'quasi-geostrophic' magnetohydrodynamic wave motions which represent a small departure from a geostrophic flow.

Magnetic instabilities are important for theories of the solar convection zone and, with this in mind, it would be most appropriate to consider a spherical shell of rotating compressible fluid. However, this geometry is difficult to treat analytically and there is justification for replacing it by a simpler (Cartesian) geometry. When geostrophic conditions prevail, the principal effect of the curvature in the boundaries of a rotating shell of fluid contained between two concentric spheres is that it alters the length, and consequently the vorticity, of a filament of fluid which is moving towards or away from the axis of rotation. This phenomenon can be modelled qualitatively in a Cartesian geometry by considering a rotating layer of fluid of average depth D which is bounded by a flat top, that is perpendicular to $\mathbf{\Omega}$, and a gently sloping bottom. We shall present such a treatment in this paper; the configuration is described in detail in § 2 [see also Pedlosky (1971), Hide (1977) and Acheson (1978*b*)]. The magnetic field is parallel with the depth contours (and perpendicular to both $\mathbf{\Omega}$ and gravity $\mathbf{g}$) and corresponds to a toroidal field in the shell. We thus expect that, when the depth decreases (increases) in the direction of $\mathbf{g}$, the fluid will exhibit behaviour which is typical of the mid-latitudes (equatorial latitudes), i.e. that region on the poleward (equatorward) side of the cylindrical surface touching the inner spherical boundary at the equator, of a thin shell of rotating, compressible, self-gravitating fluid.

We confine our attention to a parameter regime

$$N^2 \ll \mathbf{\Omega}^2, \quad V^2 \ll a^2, \quad |\mathbf{g}| \gtrsim a|\mathbf{\Omega}|, \tag{1.1}$$

$$V^2 \ll |\mathbf{\Omega}|^2 D^2, \tag{1.2}$$

which is appropriate to the (low entropy) convection zone of the sun. Here a denotes the isothermal sound speed, V the Alfvén speed and N, the Brunt–Väisälä frequency, is the frequency at which a parcel of fluid would oscillate if displaced adiabatically in the absence of rotation and magnetic field. We shall assume throughout that the fluid is statically stable, i.e. $N^2 > 0$, since we wish to describe a new instability mechanism that is distinct from simple thermal convection; but we note in passing that the eastward propagation theorem of § 4 below remains true if†

$$N^2 < 0 \quad \text{and} \quad |N|^2 \lesssim O(V^2/D^2). \tag{1.3}$$

Despite the restrictions contained in (1.1), the mathematical formulation of § 2 represents a substantial generalization of Busse's (1976) study of linearized wave motions in a Boussinesq fluid; weak compressibility is added to his model. Apart from the fact that Busse's fluid is convectively unstable at the outset (equivalent to $N^2 < 0$), the main difference between the two models concerns the viscous approximation used. We deal with this matter in the appendix. We follow Busse in considering only the component of gravity which is parallel to the temperature gradient vector.

The plan of the paper is as follows. In § 2 we describe the model used and derive the basic equations for the wave motions considered. In § 3 we present the non-diffusive theory and show that the waves are neutral. In § 4 we prove that any instabilities must propagate, and propagate eastward; we also discuss Busse's (1976) method of regarding the imaginary part of the marginal stability equation as a dispersion relation. In § 5 we examine the effect of a weakly diffusive perturbation to the adiabatic solution of § 3(*a*) and show that thermal diffusion can have a destabilizing effect. In § 6 we summarize and discuss the main results of the paper.

2. Mathematical formulation

Let us consider, then, an electrically conducting compressible fluid, of density ρ^+, which is unbounded in the x direction; the fluid is bounded by plane walls at $y = \pm\frac{1}{2}D$ and $z = D$, and by a gently sloping lower boundary $z = z_l(y)$, where $z_l(0) = 0$. (We append the boundary conditions which walls at $y = \pm\frac{1}{2}D$ require for completeness only; since we shall make a local approximation (see, for example, Goldreich & Schubert 1967; Fricke 1969) they are not strictly necessary.) The fluid rotates with constant angular velocity $\mathbf{\Omega} = \Omega\mathbf{e}_z$, where $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are unit vectors in the x, y and z directions, respectively, of the rotating Cartesian frame of reference; we shall assume throughout that $\Omega > 0$. In this frame, motion per unit volume of the fluid is described by the inviscid (see the appendix) momentum equation, conservation of mass or continuity equation, Gauss's law and the magnetic induction equation, equation of state (perfect gas law) and the equation of heat transfer (we assume there are no internal sources and ignore viscous dissipation):

$$\rho^+\left(\frac{d\mathbf{u}}{dt} + 2\mathbf{\Omega}\wedge\mathbf{u}\right) = -\nabla p^+ + \frac{\mathbf{B}^+.\nabla\mathbf{B}^+}{\mu} + \rho^+\mathbf{g}, \tag{2.1}$$

$$\frac{d\rho^+}{dt} + \rho^+\nabla.\mathbf{u} = 0, \tag{2.2}$$

$$\nabla.\mathbf{B}^+ = 0, \quad \frac{\partial\mathbf{B}^+}{\partial t} = \nabla\wedge(\mathbf{u}\wedge\mathbf{B}^+) + \eta\nabla^2\mathbf{B}^+, \tag{2.3}$$

$$p^+ = \frac{\mathbf{B}^{+2}}{2\mu} + RT^+\rho^+, \tag{2.4}$$

$$\rho^+\frac{de}{dt} - RT^+\frac{d\rho^+}{dt} = \nabla.\left(\frac{R\rho^+\kappa\nabla T^+}{\gamma-1}\right) + \epsilon_\eta. \tag{2.5}$$

† The value of N^2 corresponding to the solar convection zone is, in fact, marginally negative.

These are the usual magnetohydrodynamic equations and are amply discussed, for example, by Roberts (1967); t, $\mathbf{u} = (u, v, w)$, p^+, $\mathbf{B}^+$, T^+, $e(= (\gamma - 1)^{-1} RT^+)$ and $\mathbf{g} = g(y)\,\mathbf{e}_y$ ($g > 0$) denote, respectively, time, fluid velocity, *total* (fluid + magnetic) pressure, magnetic flux density, temperature, internal energy density and effective (including centrifugal acceleration) gravity; η, κ, μ, γ and R, all of which are assumed constant throughout the paper, denote magnetic (ohmic) diffusivity, thermal diffusivity, magnetic permeability, ratio of specific heats ($1 < \gamma < 2$) and gas constant;

$$\epsilon_\eta = \mu^{-1}\eta(\nabla \wedge \mathbf{B}^+)^2 \tag{2.6}$$

represents the rate at which magnetic energy is dissipated by ohmic heating.

Equations (1.1)–(1.5) admit the steady solution

$$\left.\begin{array}{c} \rho^+ = \rho_B(y), \quad p^+ = p_B(y), \quad T^+ = T_B(y), \\ \mathbf{B}^+ = B(y)\,\mathbf{e}_x, \quad \mathbf{u} = 0 \end{array}\right\} \tag{2.7}$$

provided

$$p_B = \frac{B^2}{2\mu} + RT_B\rho_B, \tag{2.8}$$

$$-dp_B/dy + \rho_B g = 0 \tag{2.9}$$

and, strictly,

$$\frac{d^2B}{dy^2} = 0, \quad \frac{R\kappa}{\gamma - 1}\frac{d}{dy}\left(\rho_B\frac{dT_B}{dy}\right) + \frac{\eta}{\mu}\left(\frac{dB}{dy}\right)^2 = 0. \tag{2.10}$$

However, suppose L and λ are the length scales associated, respectively, with the basic state (2.7) and its subsequent perturbation; the diffusion times L^2/κ, L^2/η over which the steady state will evolve are very greatly in excess of the diffusion times λ^2/κ, λ^2/η associated with the disturbances considered in this paper because we make the 'local' or 'narrow gap' approximation

$$\lambda \ll L. \tag{2.11}$$

Hence, (2.10) may be ignored in practice; we are than free to prescribe $B(y)$ and $T_B(y)$ at will. (In fact, the magnitude of the steady state contribution to the energy equation, when suitably non-dimensionalized, is quite clearly smaller than even the last term – the least important term – of equation (2.39) below.)

When a geostrophic balance obtains, viscosity will be important in Ekman boundary layers, of thickness $E^{\frac{1}{2}}D$, at the top and bottom of the fluid, where ν is the kinematic viscosity, the Ekman number is defined by

$$E = \nu/\Omega D^2 \tag{2.12}$$

and $E \ll 1$. The effect of these boundary layers (Greenspan 1968) is to create, at the upper and lower boundaries, a small suction velocity

$$\frac{1}{2}\left(\frac{\nu}{\boldsymbol{\Omega}.\mathbf{n}}\right)^{\frac{1}{2}}\mathbf{n}.\nabla \wedge \mathbf{u}_\parallel, \tag{2.13}$$

where $\mathbf{u}_\parallel$ is the velocity parallel to the boundary and $\mathbf{n}$ is the unit normal to it. If $\theta = dz_l/dy$ is the slope of the bottom boundary and $\theta \ll 1$,

$$\begin{aligned} \mathbf{n} &= \nabla(z - z_l(y)) = (1 + \theta^2)^{-\frac{1}{2}}(0, -\theta, 1) \\ &\simeq (0, -\theta, 1); \end{aligned} \tag{2.14}$$

also, since $\theta \ll 1$, $\cos\theta \simeq 1$ and we can replace $\mathbf{u}_\parallel$ by $(u, v, 0)$. We therefore write the Ekman suction velocity as

$$w_s = \tfrac{1}{2}(\nu/\Omega)^{\frac{1}{2}}(\partial v/\partial x - \partial u/\partial y), \tag{2.15}$$

which is of order $E^{\frac{1}{2}}$ compared to the flow in the interior of the fluid.

Accordingly, we now write

$$\left.\begin{aligned} \rho^+ = \rho_B + \rho, \quad p^+ = p_B + p, \quad T^+ = T_B + T, \\ \mathbf{B}^+ = B\mathbf{e}_x + \mathbf{b}, \quad \mathbf{u} = (u, v, w) \end{aligned}\right\} \tag{2.16}$$

where $\mathbf{b} = (b_x, b_y, b_z)$, and perturb the steady solution (2.7) subject to the boundary conditions

$$\left.\begin{aligned} w &= v\frac{dz_l}{dy} + w_s \quad \text{at} \quad z = z_l(y), \\ w &= -w_s \quad \text{at} \quad z = D, \end{aligned}\right\} \tag{2.17}$$

$$v = 0, T = 0 \quad \text{at} \quad y = \pm\tfrac{1}{2}D. \tag{2.18}$$

Assuming that $\mathbf{u}$, p/p_B, ρ/ρ_B, etc. are small, substituting (2.16) into equations (2.1)–(2.5) and neglecting products of small quantities, we obtain the following set of linearized equations:

$$\frac{\partial \mathbf{u}}{\partial t} + 2\Omega \wedge \mathbf{u} = \frac{-1}{\rho_B}\nabla p + \frac{B}{\mu\rho_B}\frac{\partial \mathbf{b}}{\partial x} + \frac{B'}{\mu\rho_B}b_y\mathbf{e}_x + \frac{\rho g}{\rho_B}\mathbf{e}_y, \tag{2.19}$$

$$\frac{1}{\rho_B}\frac{\partial \rho}{\partial t} + \frac{\rho_B'}{\rho_B} + \nabla.\mathbf{u} = 0, \tag{2.20}$$

$$\nabla.\mathbf{b} = 0, \tag{2.21}$$

$$\frac{\partial \mathbf{b}}{\partial t} = B\frac{\partial \mathbf{u}}{\partial x} - \mathbf{e}_x\nabla.(B\mathbf{u}) + \eta\nabla^2\mathbf{b}, \tag{2.22}$$

$$p = R(T\rho_B + \rho T_B) + \frac{Bbx}{\mu}, \tag{2.23}$$

and

$$\frac{\partial}{\partial t}\left(\frac{T}{T_B} - (\gamma-1)\frac{\rho}{\rho_B}\right) = T_B^{-1}\kappa\left(\nabla^2 T + \frac{\rho_B'}{\rho_B}\frac{\partial T}{\partial y}\right) + \gamma\frac{N^2 v}{g} - \frac{2\eta B'(\gamma-1)}{\mu\rho_B a_0^2}\left(\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}\right), \tag{2.24}$$

where a prime denotes differentiation with respect to y. We define

$$V(y) \equiv B/(\mu\rho_B)^{\frac{1}{2}} \tag{2.25}$$

$$a_0(y) \equiv (RT_B)^{\frac{1}{2}}, \tag{2.26}$$

$$N(y) \equiv \left\{-\frac{g}{\gamma}\frac{d}{dy}[\log(T_B\rho_B^{1-\gamma})]\right\}^{\frac{1}{2}}. \tag{2.27}$$

These are, respectively, the Alfvén speed, isothermal sound speed and Brunt–Väisälä frequency. We assume [cf. equation (1.1)] that

$$|V|^2 \ll |a_0|^2. \tag{2.28}$$

Then, with a very small error (of fourth order in the expansion parameter; see below), we can say that p_B in (2.8) and (2.9) is simply the fluid pressure in the basic state and write (2.27) in its more familiar form

$$N = \left\{-gT_B^{-1}\left(\frac{dT_B}{dy} - \left.\frac{dT}{dy}\right|_{\mathrm{ad}}\right)\right\}^{\frac{1}{2}}, \tag{2.29}$$

where

$$\left.\frac{dT}{dy}\right|_{\mathrm{ad}} = \frac{(\gamma-1)g}{\gamma R} \tag{2.30}$$

is the adiabatic temperature gradient. We shall assume throughout that the temperature gradient is sub-adiabatic (N real) and close to the adiabatic one, because we must satisfy (1.1), of which we remind ourselves here:

$$N^2 \ll \mathbf{\Omega}^2, \quad g \gtrsim a_0 \Omega. \tag{2.31}$$

We now non-dimensionalize t, (x, y, z), $\mathbf{u}$, $\mathbf{b}$, p, ρ, T, η, κ and z_l by, respectively, τ, D, U, $|B|\,\tau UD^{-1}$, $|\rho_B|\,U\Omega D$, $|\rho_B|\,U\Omega D\,|a_0|^{-2}$, $|T_B|\,U\Omega DR^{-1}$, $D^2\tau^{-1}$, $D^2\tau^{-1}$ and h (see figure 1); also, the basic state functions B, ρ_B, a_0, etc. vary by a factor of order unity over a length scale L and

$$\left|\frac{L\rho_B'}{\rho_B}\right| \sim \left|\frac{LB'}{B}\right| \sim \left|\frac{La_0'}{a_0}\right| \sim O(1). \tag{2.32}$$

Thus, in non-dimensional form, equations (2.19)–(2.24) become

$$\frac{1}{\Omega T}\frac{\partial \bar{\mathbf{u}}}{\partial \bar{t}} - 2\bar{v}\mathbf{e}_x + 2\bar{u}\mathbf{e}_y = \frac{-1}{\bar{\rho}_B}\bar{\nabla}\bar{p} + \frac{|g|\,D}{|a_0|^2}\cdot\frac{\bar{g}}{\bar{\rho}_B}\bar{\rho}\mathbf{e}_y + \frac{|V|^2}{\Omega^2 D^2}\cdot\Omega\tau\cdot\frac{\bar{B}}{\bar{\rho}_B}\left[\frac{\partial\bar{\mathbf{b}}}{\partial\bar{x}} + \frac{D}{L}\cdot\frac{\bar{B}'}{\bar{B}}\bar{b}_y\,\mathbf{e}_x\right], \tag{2.33}$$

$$\frac{1}{\Omega\tau}\cdot\frac{\Omega^2 D^2}{|a_0|^2}\frac{\partial\bar{\rho}}{\partial\bar{t}} + \frac{D}{L}\cdot\frac{\bar{\rho}_B'}{\bar{\rho}_B}\bar{v} + \bar{\nabla}.\bar{\mathbf{u}} = 0, \tag{2.34}$$

$$\bar{\nabla}.\bar{\mathbf{b}} = 0, \tag{2.35}$$

$$\frac{\partial\bar{b}_x}{\partial\bar{t}} = -\bar{B}\left(\frac{\partial\bar{v}}{\partial\bar{y}} + \frac{\partial\bar{w}}{\partial\bar{z}} + \frac{D}{L}\cdot\frac{\bar{B}'}{\bar{B}}\bar{v}\right) + \bar{\eta}\bar{\nabla}^2\bar{b}_x, \tag{2.36}$$

$$\frac{\partial}{\partial\bar{t}}(\bar{b}_y, \bar{b}_z) = \bar{B}\frac{\partial}{\partial\bar{x}}(\bar{v}, \bar{w}) + \bar{\eta}\nabla^2(\bar{b}_y, \bar{b}_z), \tag{2.37}$$

$$\bar{p} = \bar{\rho}_B\bar{T} + \bar{T}_B\bar{\rho} + \frac{|V|^2}{\Omega^2 D^2}\cdot\Omega\tau.\bar{B}\bar{b}_x \tag{2.38}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\bar{T}}{\bar{T}_B} - (\gamma-1)\frac{\bar{\rho}}{\bar{\rho}_B}\right) = T_B^{-1}\bar{\kappa}\left[\bar{\nabla}^2\bar{T} + \frac{D}{L}\cdot\frac{\bar{\rho}_B'}{\bar{\rho}_B}\frac{\partial\bar{T}}{\partial\bar{y}}\right] + \frac{|a_0|^2}{|g|\,D}\cdot\frac{|N|^2}{\Omega^2}\cdot\Omega\tau\cdot\gamma\frac{\bar{N}^2}{\bar{g}}\bar{v} - \frac{2\bar{\eta}\bar{B}'(\gamma-1)}{\bar{\rho}_B\bar{T}_B}\frac{|V|^2}{\Omega^2 D^2}\cdot\frac{D}{L}\cdot\Omega\tau\left(\frac{\partial\bar{b}_y}{\partial\bar{x}} - \frac{\partial\bar{b}_x}{\partial\bar{y}}\right), \tag{2.39}$$

while the boundary condition (2.17) becomes

$$\left.\begin{aligned}\bar{w} &= \frac{h}{D}\bar{v}\frac{d\bar{z}_l}{d\bar{y}}+\tfrac{1}{2}E^{\frac{1}{2}}\left(\frac{\partial\bar{v}}{\partial\bar{x}}-\frac{\partial\bar{u}}{\partial\bar{y}}\right) \quad \text{at} \quad \bar{z}=\frac{h}{D}\bar{z}_l,\\ \bar{w} &= -\tfrac{1}{2}E^{\frac{1}{2}}\left(\frac{\partial\bar{y}}{\partial\bar{x}}-\frac{\partial\bar{u}}{\partial\bar{y}}\right) \quad \text{at} \quad \bar{z}=1.\end{aligned}\right\} \tag{2.40}$$

Here, an overbar denotes a dimensionless quantity; $\bar{\eta}=\eta\tau D^{-2}$, $\bar{\kappa}=\kappa\tau D^{-2}$. In view of (2.11) with $\lambda\sim D$, it is important to note that the functions $\bar{B}$, $\bar{\rho}_B$, $\bar{T}_B$, $\bar{g}$ and $\bar{N}$ which appear in equations (2.33)–(2.39) depend on the 'slow' variable (see (2.41) below) $DL^{-1}\bar{y}$ and a prime denotes differentiation with respect to that argument. We now choose

$$\delta = hD^{-1}, \quad \Gamma^0\delta = DL^{-1}, \quad E^0\delta = E^{\frac{1}{2}}, \tag{2.41}$$

$$(\Omega\tau)^{-1} = G^0\delta, \quad |v|\,\Omega^{-1}D^{-1} = H^0\delta, \quad |g|\,D\,|a_0|^{-2} = K^0\delta, \tag{2.42}$$

$$\Omega D\,|a_0|^{-1} = O(\delta), \quad |N|\,\Omega^{-1} = O(\delta), \tag{2.43}$$

where $\delta \ll 1$ and Γ^0, E^0, G^0, H^0 and K^0 are $O(1)$ constants. Thus, in the momentum equation, geostrophy obtains at leading order while Lorentz forces contribute at the next order; and, with $\lambda\sim D$, (2.11), (2.28) and (2.31) are well satisfied. In particular, $|V|^2\,|a_0|^{-2} = O(\delta^4)$. The above choice of parameters filters fast acoustic modes (sound waves) out of the wave spectrum, since the temporal term in the continuity equation is too small to play a part in the subsequent analysis. The choice does not, however, filter out the slow acoustic modes (or internal gravity waves; see Holton 1972).

We look for solutions of (2.33)–(2.40) in the form

$$\bar{\mathbf{u}} = \bar{\mathbf{u}}_0+\delta\bar{\mathbf{u}}_1+O(\delta^2) \tag{2.44}$$

and so on. Dropping, for convenience, overbars on all except the basic state variables, we obtain from (2.33), (2.35) and (2.40) the zeroth order (in δ) equations

$$-2v_0 = \frac{-1}{\bar{\rho}_B}\frac{\partial p_0}{\partial x}, \quad 2u_0 = \frac{-1}{\bar{\rho}_B}\frac{\partial p_0}{\partial y}, \tag{2.45a,b}$$

$$\frac{\partial p_0}{\partial z} = 0, \tag{2.46}$$

$$\nabla\cdot\mathbf{b}_0 = 0 \tag{2.47}$$

and the zeroth-order boundary condition

$$w_0 = 0 \quad \text{at} \quad z = 0, 1. \tag{2.48}$$

Cross differentiating (2.45) gives

$$\frac{\partial}{\partial x}(\rho_B u_0)+\frac{\partial}{\partial y}(\rho_B v_0) = 0. \tag{2.49}$$

The most illuminating form of (2.34) is actually one in which no assumption has yet been made concerning the rate at which ρ_B varies with y:

$$\frac{\partial}{\partial x}[\rho_B(u_0+\delta u_1)]+\frac{\partial}{\partial y}[\rho_B(v_0+\delta v_1)]+\frac{\partial}{\partial z}[\rho_B(w_0+\delta w_1)] = O(\delta^2). \tag{2.50}$$

At zeroth order in δ this gives

$$\frac{\partial}{\partial z}[\rho_B w_0] = 0, \tag{2.51}$$

on using (2.49). At first order in δ, (2.50) gives

$$\frac{\partial}{\partial x}(\rho_B u_1)+\frac{\partial}{\partial y}(\rho_B v_1)+\frac{\partial}{\partial z}(\rho_B w_1) = 0, \tag{2.52}$$

while (2.33) leads to

$$\left.\begin{aligned} 2u_1 &= -G_0^0\frac{\partial v_0}{\partial t}-\bar{\rho}_B^{-1}\frac{\partial p_1}{\partial y}+\frac{K^0\bar{g}}{\bar{\rho}_B}\rho_0+\frac{(H^0)^2}{G_0}\frac{\bar{B}}{\bar{\rho}_B}\frac{\partial b_{y_0}}{\partial x}, \\ 2v_1 &= G^0\frac{\partial u_0}{\partial t}+\bar{\rho}_B^{-1}\frac{\partial p_1}{\partial x}-\frac{(H^0)^2}{G^0}\frac{\bar{B}}{\bar{\rho}_B}\frac{\partial b_{z_0}}{\partial x}. \end{aligned}\right\} \tag{2.53}$$

If we *now* make the local approximation that ρ_B varies slowly and use (2.48), we obtain $w_0 = 0$ and, from (2.52),

$$\frac{\partial w_1}{\partial z} = -\frac{\partial u_1}{\partial x}-\frac{\partial v_1}{\partial y}. \tag{2.54}$$

Clearly p_0, u_0, v_0 and w_0 are all independent of z; the fluid motions are two-dimensional to leading order. We assume that $\mathbf{b}_0$, ρ_0 and T_0 are also independent of z.

An alternative derivation of (2.54) has been given by Gibbons (1977). It is an important point that the fluid is non-divergent (at this order) but not incompressible: according to (2.57) below, the density is allowed to respond to changes in pressure. The equation of state for a Boussinesq fluid allows the density to respond only to changes in temperature. Thus the work done by compression is an energy source for instability that is not incorporated into Busse's (1976) model (see § 5 below).

Substituting the expressions (2.53) for u_1 and v_1 into (2.54) gives a right-hand side which is independent of z; using (2.47) to eliminate b_{x_0}, integrating (2.54) between $z = 0$ and $z = 1$ and applying the first-order boundary condition

$$w = \frac{dz_l}{dy}+\tfrac{1}{2}E^{0\frac{1}{2}}\left(\frac{\partial v_0}{\partial x}-\frac{\partial u_0}{\partial y}\right) \quad \text{at} \quad z = 0,$$

$$w = -\tfrac{1}{2}E^{0\frac{1}{2}}\left(\frac{\partial v_0}{\partial x}-\frac{\partial u_0}{\partial y}\right) \quad \text{at} \quad z = 1,$$

gives, restoring overbars:

$$G^0\frac{\partial}{\partial \bar{t}}\left(\frac{\partial \bar{v}_0}{\partial \bar{x}}-\frac{\partial \bar{u}_0}{\partial \bar{y}}\right)-\frac{K^0\bar{g}}{\bar{\rho}_B}\frac{\partial \bar{\rho}_0}{\partial \bar{x}}-\frac{(H^0)^2}{G^0}\frac{\bar{B}}{\bar{\rho}_B}\left(\frac{\partial^2}{\partial \bar{x}^2}+\frac{\partial^2}{\partial \bar{y}^2}\right)b_{\bar{y}_0} + 2\left[\bar{v}_0\frac{d\bar{z}_l}{d\bar{y}}+E^{0\frac{1}{2}}\left(\frac{\partial \bar{v}_0}{\partial \bar{x}}-\frac{\partial \bar{u}_0}{\partial \bar{y}}\right)\right] = 0.$$

The dimensional form of this equation is

$$-\frac{\partial}{\partial t}\left(\frac{\partial v_0}{\partial x}-\frac{\partial u_0}{\partial y}\right)+\frac{g}{\rho_B}\frac{\partial \rho_0}{\partial x}+\frac{B}{\mu\rho_B}\nabla_h^2 b_{y_0} = 2\Omega\left\{\frac{v_0}{D}\frac{dz_l}{dy}+E^{\frac{1}{2}}\left(\frac{\partial v_0}{\partial x}-\frac{\partial u_0}{\partial y}\right)\right\}, \tag{2.55}$$

where $\nabla_h^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$. Henceforward, it is most convenient to work with dimensional quantities; in these terms, the zeroth-order versions of (2.37)–(2.39) give (recalling that zeroth-order functions are independent of z):

$$\left(\frac{\partial}{\partial t} - \eta\nabla_h^2\right) b_{y_0} = B\frac{\partial v_0}{\partial x}, \tag{2.56}$$

$$p_0 = R(\rho_B T_0 + T_B \rho_0) \tag{2.57}$$

and

$$\frac{\partial}{\partial t}\left[\frac{T_0}{T_B} - \frac{(\gamma-1)\rho_0}{\rho_B}\right] = T_B^{-1}\kappa\nabla_h^2 T_0 + \gamma N^2 v_0 g^{-1}. \tag{2.58}$$

Fquations (2.55)–(2.58), together with the dimensional forms of (2.45*a*) and (2.49), are six equations for the six unknowns u_0, v_0, b_{y_0}, p_0, ρ_0 and T_0. Since the steady-state functions which appear in these equations hardly vary across the fluid (because $D \ll L$), we can assume that they are constant; solutions satisfying the boundary condition (2.18) at zeroth order are then

$$v_0, T_0, \rho_0, b_{y_0}, p_0 \propto \mathrm{Re}\{\exp[i(kx-\omega t)]\sin[m(y+\tfrac{1}{2}D)]\}, \tag{2.59}$$

$$u_0 \propto \mathrm{Re}\{\exp[i(kx-\omega t)]\cos[m(y+\tfrac{1}{2}D)]\}, \tag{2.60}$$

where Re denotes real part, provided only that m is a multiple of πD^{-1} and the following consistency condition is satisfied:

$$\frac{\gamma N^2 k^2}{\gamma\omega + i\kappa s^2} + \frac{V^2 k^2 s^2}{w + i\eta s^2} - \frac{2\Omega k}{D}\frac{dz_l}{dy} - s^2(\omega + 2\Omega i E^{\frac{1}{2}}) = \frac{-2\Omega k g}{a_0^2}\cdot\frac{\omega + i\kappa s^2}{\gamma\omega + i\kappa s^2}, \tag{2.61}$$

where

$$s^2 = k^2 + m^2. \tag{2.62}$$

The novel term in (2.61) is the one on the right-hand side which is due to compressibility. If this term is removed, (2.61) reduces to Busse's (1976) equation (12) when we let $\gamma \to 1$ and alter details (change ω to $-\omega$, s^2 to his a^2, etc.); in place of our N^2, Busse's model has a negative buoyancy parameter which drives convection by virtue of an adverse (equivalent to superadiabatic in our terms) temperature gradient. Consistency condition (2.61) forms a dispersion relation for waves in the fluid and will be used hereafter to determine its stability.

3. Non-diffusive modes

We shall discuss non-diffusive modes in two, quite different, limits; first of all, we recall a theorem due to Ertel which (Greenspan 1968), although it does not apply to an electrically conducting fluid, will be extremely useful in the discussion that follows. The theorem says that, in the absence of viscosity, the potential vorticity

$$\Phi \equiv \frac{(\mathbf{J} + 2\mathbf{\Omega}).\nabla\Lambda}{\rho^+} \tag{3.1}$$

of a fluid rotating with angular velocity $\mathbf{\Omega}$ satisfies

$$\frac{d\Phi}{dt} = (\rho^+)^{-1}\nabla\Lambda.\nabla p^+ \wedge \nabla(\rho^+)^{-1}, \tag{3.2}$$

where $\mathbf{J} = \nabla \wedge \mathbf{u}$ is the vorticity relative to the rotating frame and Λ is any scalar quantity conserved by individual fluid elements throughout their motion. For the geostrophic flows considered here, $\Lambda = \chi^{-1}(z-z_l)$ is such a quantity, where χ is the length of a filament of fluid (Pedlosky 1971); and $\mathbf{J}$ has only one component, parallel to $\mathbf{\Omega}$. Also, if the fluid is either homogeneous (ρ^+ constant), isothermal ($p^+ \propto \rho^+$) or isentropic ($p^+ \propto \rho^{+\gamma}$), the only situations to be considered in § 3 (*a*) and § 3 (*b*), then the right-hand side of (3.2) vanishes. Thus the potential vorticity is conserved:

$$(J+2\Omega)/\rho\chi = \text{constant}. \tag{3.3}$$

(*a*) $E = \eta = \kappa = 0$: *adiabatic modes*

When $E = \eta = \kappa = 0$, the fluid exchanges no heat with its surroundings and (2.61) reduces to that which we would have obtained by dispensing with the energy equation and using the isentropic law $p \propto \rho^\gamma$ as the equation of state:

$$\omega^2 + B_* \omega - V^2k^2 - N^2k^2s^{-2} = 0, \tag{3.4}$$

where

$$\beta_* \equiv \frac{2\Omega k}{s^2}\left(\frac{1}{D}\frac{dz_l}{dy} - \frac{g}{a_*^2}\right) \tag{3.5}$$

will be called the adiabatic Rossby frequency and $a_* = \gamma^{\frac{1}{2}}a_0$ is the adiabatic sound speed. (A knowledge of Rossby waves in an incompressible fluid is assumed here but may be gleaned from Holton 1972.) If we replace β_* by the incompressible Rossby frequency

$$\beta_{\text{inc}} \equiv \frac{2\Omega k}{s^2 D}\frac{dz_l}{dy}, \tag{3.6}$$

the dispersion relation thus obtained,

$$\omega^2 + \beta_{\text{inc}}\omega - V^2k^2 - N^2k^2s^{-2} = 0, \tag{3.7}$$

agrees with one previously obtained (Gibbons 1975) for a Boussinesq fluid when N is replaced by the incompressible Brunt–Väisälä frequency $\{g\rho_B^{-1}\rho_B'\}^{\frac{1}{2}}$; if also $N = 0$, it agrees with that obtained by Hide (1966). Other limits are Alfvén waves ($\Omega = N = 0$) and internal gravity waves ($\Omega = V = 0$).

When $V = N = 0$, the pure Rossby wave

$$\frac{\omega}{k} = -\frac{\beta_{\text{inc}}}{k} = -\frac{2\Omega}{s^2 D}\frac{dz_l}{dy} \tag{3.8}$$

propagates so that the deep fluid is on the left, in the positive or negative x direction according as $dz_l/dy < 0$ or $dz_l/dy > 0$. As Acheson (1978*b*) has stressed recently, there is no concept of eastward or westward propagation here: the wave is totally insensitive to where in the x, y plane the rotation axis is located. The concept reappears as soon as we introduce compressibility, however: we associate eastward propagation with the positive x direction, that of $\mathbf{g} \wedge \mathbf{\Omega}$. Suppose now that $dz_l/dy > 0$. Then the mechanism by which an incompressible (pure, i.e. not modified by magnetic and stratification effects) Rossby wave propagates westward can be understood in terms of Acheson & Hide's (1973) argument, for a homogeneous fluid, involving diagrams of the instantaneous velocity profile. This argument, of which a detailed presentation would distract us unnecessarily here, is essentially based on the observation that (when

$dz_l/dy > 0$) an outward moving filament of fluid ($v < 0$) increases in length and gains vorticity, while an inward moving filament decreases in length and loses vorticity, as can be seen by putting ρ = constant in (3.3). When the fluid is compressible, however, the expansion (compression) of a filament of fluid as it moves outward (inward) causes a decrease (increase) in its density ρ which, by virtue of (3.3), is responsible for a decrease (increase) in vorticity that counteracts the increase (decrease) due to topographic effects and, by simply reversing Acheson & Hide's argument, gives the Rossby wave a propensity for eastward propagation. Putting $V = N = 0$ in (3.4), we can quantify this balance between opposed vortiginous effects by saying that the direction of propagation of the pure Rossby wave depends upon the sign of

$$\beta_* \equiv \frac{1}{D}\frac{dz_l}{dy} - \frac{g}{a_*^2}. \tag{3.9}$$

When $dz_\rho/dy < 0$, of course, the topographic and thermodynamic vorticity effects reinforce, rather than oppose, each other and the pure Rossby wave always propagates eastwards.

Let us now define

$$J_* \equiv \beta_*^{-2}(V^2k^2 + N^2k^2s^{-2}). \tag{3.10}$$

The solutions of (3.4) are thus given by

$$\omega = -\tfrac{1}{2}\beta_*\{1 \pm (1+4J_*)^{\frac{1}{2}}\} \tag{3.11}$$

so that the roots ω are always real and are such that the waves will always propagate in opposite directions. (The fact that the waves are dynamically neutral extends to the case of a destabilizing entropy gradient $N^2 < 0$ if (1.3) is satisfied and the wavenumbers k, m are of order D^{-1}.) Henceforward, the solution (3.11) for which the positive square-root sign is taken will be known as the *fast* wave, while that for which the negative sign is taken will be known as the *slow* wave. We choose this terminology because the frequencies of the wave pair are widely disparate when (cf. Hide 1966)

$$|J_*| \ll 1. \tag{3.12}$$

The solutions of (3.11) are then given approximately by

$$\omega_+ \simeq -\beta_*, \quad \omega_- \simeq \beta_* J_*, \tag{3.13}$$

and $|\omega_-| \ll |\omega_+|$. The first of (3.13) corresponds to an almost pure Rossby wave, for which we expect magnetic effects to be unimportant; the second is a slow magnetohydrodynamic wave of the kind that has been important in studies of magnetic buoyancy.

(*b*) $E = \eta = 0$, $\kappa = \infty$: *isothermal modes*

When $\kappa = \infty$, the fluid exchanges heat isothermally with its surroundings and (2.59) reduces to that which we would have obtained by dispensing with the energy equation and using the isothermal law $p \propto \rho$ as the equation of state:

$$\omega^2 + \beta_0\omega - V^2k^2 = 0, \tag{3.14}$$

where

$$\beta_0 \equiv \frac{2\Omega k}{s^2}\left(\frac{1}{D}\frac{dz_l}{dy} - \frac{g}{a_0^2}\right) \tag{3.15}$$

will be called the isothermal Rossby frequency. Comparing (3.14) with (3.4) we see that letting $\kappa \to \infty$ has the same effect as letting $N \to 0$: it completely annihilates adiabatic buoyancy. Once again, the roots

$$\omega = -\tfrac{1}{2}\bar{\beta}_0\{1 \pm (1+4J_0)^{\frac{1}{2}}\} \tag{3.16}$$

of (3.14) correspond to a pair of dynamically neutral waves which propagate in opposite directions; and the precise way in which the direction of propagation of the pure Rossby wave ($V = 0$) depends upon the sign of

$$\bar{\beta}_0 \equiv \frac{1}{D}\frac{dz_l}{dy} - \frac{g}{a_0^2} \tag{3.17}$$

can be understood by analogy with the adiabatic case. Here, $J_0 \equiv \bar{\beta}_0^{-2} V^2 k^2$. In the rapid rotation limit $|J_0| \ll 1$ the waves are given approximately by

$$\omega_+ \simeq -\bar{\beta}_0, \quad \omega_- \simeq \bar{\beta}_0 J_0. \tag{3.18}$$

(c) *Vorticity equation for pure Rossby modes*

When there is no magnetic field and no entropy gradient, equation (2.55) can be reduced to the form

$$\frac{\partial}{\partial t}\left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}\right) + 2\Omega v_0\left[\frac{1}{D}\frac{dz_l}{dy} - \frac{g}{a^2}\right] = 0, \tag{3.19}$$

where a is the appropriate sound speed, for both (adiabatic and isothermal) pure Rossby waves. This equation enables us to see directly how the net change in vorticity depends upon the sign of $\bar{\beta}$; for example, if $\bar{\beta} > 0$, vorticity increases for an outward ($v_0 < 0$) moving filament. Equation (3.19) can also be obtained (see Gibbons 1977) by linearizing (3.3) subject to the appropriate relation between p and ρ.

In § 5, we will solve (2.61) by subjecting (3.4) to a weakly diffusive perturbation such that

$$\left|\frac{\Omega E^{\frac{1}{2}}}{\omega}\right|, \quad \left|\frac{\eta s^2}{\omega}\right|, \quad \left|\frac{\kappa s^2}{\omega}\right| \ll 1. \tag{3.20}$$

We will therefore assume that a typical diffusion time, associated with any of the three diffusions, is very much longer than a period of oscillation.

4. Propagation results: marginal stability

Before proceeding to the weakly diffusive analyses mentioned at the end of the last section, we derive a result which is independent of the strengths of the diffusions, namely that all instabilities of the fluid must take the form of eastward propagating, amplifying waves.

Let us define the real and imaginary parts of ω by

$$\omega = \omega_{\text{Re}} + i\omega_{\text{Im}} \tag{4.1}$$

and the reduced (since $\gamma > 1$) thermal diffusivity κ' by

$$\kappa' \equiv \kappa/\gamma; \tag{4.2}$$

$\omega_{\mathrm{Im}} > 0$ corresponds to instability. The real and imaginary parts of (2.61) are given by

$$N^2k^2 + V^2k^2s^2\frac{[\omega_{\mathrm{Re}}^2+(\omega_{\mathrm{Im}}+\kappa' s^2)(\omega_{\mathrm{Im}}+\eta s^2)]}{\omega_{\mathrm{Re}}^2+(\omega_{\mathrm{Im}}+\eta s^2)^2} - \frac{2\Omega k\omega_{\mathrm{Re}}}{D}\frac{dz_l}{dy}$$
$$-s^2\{\omega_{\mathrm{Re}}^2-(\omega_{\mathrm{Im}}+2\Omega E^{\frac{1}{2}})(\omega_{\mathrm{Im}}+\kappa' s^2)\} = -\frac{2\Omega gk\omega_{\mathrm{Re}}}{a_*^2} \tag{4.3}$$

and

$$\frac{V^2k^2s^4\omega_{\mathrm{Re}}(\kappa'-\eta)}{\omega_{\mathrm{Re}}^2+(\omega_{\mathrm{Im}}+\eta s^2)^2} - \frac{2\Omega k}{D}\frac{dz_l}{dy}(\omega_{\mathrm{Im}}+\kappa' s^2) - s^2[2\omega_{\mathrm{Im}}+2\Omega E^{\frac{1}{2}}+\kappa' s^2]\,\omega_{\mathrm{Re}}$$
$$= -\frac{2\Omega kg}{a_*^2}(\omega_{\mathrm{Im}}+\kappa s^2). \tag{4.4}$$

Multiplying (4.3) by $\omega_{\mathrm{Im}}+\kappa' s^2$, multiplying (4.4) by ω_{Re} and subtracting, we therefore obtain

$$N^2k^2(\omega_{\mathrm{Im}}+\kappa' s^2) + \frac{V^2k^2s^2(\omega_{\mathrm{Im}}+\eta s^2)\{(\omega_{\mathrm{Im}}+\kappa' s^2)^2+\omega_{\mathrm{Re}}^2\}}{\omega_{\mathrm{Re}}^2+(\omega_{\mathrm{Im}}+\eta s^2)^2}$$
$$+s^2(\omega_{\mathrm{Im}}+2\Omega E^{\frac{1}{2}})\{(\omega_{\mathrm{Im}}+\kappa' s^2)^2+\omega_{\mathrm{Re}}^2\} = \frac{2\Omega gk\omega_{\mathrm{Re}}\,\kappa' s^2(\gamma-1)}{a_*^2}. \tag{4.5}$$

Suppose that $\omega_{\mathrm{Re}} = 0$; then the right-hand side of (4.5) vanishes and the equation which remains can only be satisfied if $\omega_{\mathrm{Im}} < 0$, corresponding to decay. Thus any instability must propagate; there is no stationary instability, a result which we could also have derived directly from (4.3). Suppose now that $\omega_{\mathrm{Im}} > 0$; then the left-hand side of (4.5) is positive. Hence, since $\gamma > 1$, ω_{Re}/k must be greater than zero, so that the diiection of propagation is eastward. Even if N^2 is negative, this result clearly remains true as long as $|N|^2 < V^2s^2$; with $|k| \sim D^{-1}$, $|m| \sim D^{-1}$, this condition is equivalent to (1.3).

The propensity for eastward propagation is so strong that it is quite independent of both the gradient of depth dz_l/dy and the relative magnitudes of the diffusivities involved. It is clearly due to compressibility: the right-hand side of (4.5) would vanish under the Boussinesq approximation. Indeed, there can be no instability at all without compressibility because the vanishing of the right-hand side of (4.5) requires ω_{Im} to be negative.

The method we have used to obtain information about the direction of propagation is thoroughly dependable; but we now proceed to explain why the method used by Busse (1976) must be regarded with caution. At marginal stability, $\omega_{\mathrm{Im}} = 0$; writing $\omega = \omega_{\mathrm{Re}}$, (4.3) and (4.4) become

$$\omega^2+\beta_*\omega - N^2k^2s^{-2} - 2\Omega E^{\frac{1}{2}}\kappa' s^2 - V^2k^2(\omega^2+\eta^2s^4)^{-1}(\omega^2+\eta\kappa' s^4) = 0 \tag{4.6}$$

and

$$\omega^2(1+P)+\beta_0\omega - \frac{V^2k^2\omega^2(1-q)}{\omega^2+\eta^2s^4} = 0, \tag{4.7}$$

where we define

$$q \equiv \eta/\kappa', \quad P \equiv \frac{2\Omega E^{\frac{1}{2}}}{\kappa' s^2}. \tag{4.8}$$

Since ω can in theory be eliminated from equations (4.6) and (4.7), which should be regarded as a dispersion relation? Suppose, following Busse (1976), we take the imaginary part of the marginal stability equation, i.e. (4.7), which is identical in form to Busse's equation (13*a*). Again following Busse, we observe that (4.7) reduces to a quadratic in ω in the two partially overlapping limits

$$|\omega(1+P)| \ll |\beta_0|, \quad |\omega| \gg |\eta s^2|;$$

solving (4.7) in the first limit gives

$$\omega = \frac{V^2k^2(1-q)}{2\beta_0} \pm \frac{1}{2}\left\{\frac{(V^2k^2)^2}{\beta_0^2}(1-q)^2 - 4\eta^2 s^4\right\}^{\frac{1}{2}}; \tag{4.9}$$

and when

$$\left|\frac{V^2k^2}{\beta_0}\right| \gg |\eta s^2|, \tag{4.10}$$

one of the roots is approximately

$$\frac{V^2k^2(1-q)}{\beta_0}. \tag{4.11}$$

From (4.11), we see that (4.10) is equivalent to $|\omega| \gg |\eta s^2|$ in this case.

It now appears that the direction of propagation of a slow wave in the x direction depends on whether $q \lessgtr 1$, contradicting the general result obtained earlier. The flaw in the reasoning of the last paragraph arises because we assume that q and P are of order unity. Let us inspect (4.6) and (4.7) a little more closely. If we put $E = \eta = \kappa = 0$ in (4.6) we obtain equation (3.4), the dispersion relation for adiabatic modes. If we put $E = \eta = 0$, $\kappa = \infty$ in (4.7) we obtain equation (3.14), the dispersion relation for isothermal modes. Thus, when $|\omega| \gg |\eta s^2|$ and diffusion is weak, we expect (4.6) to be close, in some sense, to the adiabatic solution and (4.7) to be close to the isothermal one. A little thought reveals that the only sense in which (4.6) can be close to the adiabatic solution is that $|\kappa' s^2| \ll |\omega|$ (cf. (3.20) – the thermal relaxation time is very long compared to a period of oscillation); the only sense in which (4.7) can be close to the isothermal solution (3.14) is that $|\omega| \ll |\kappa' s^2|$ (the thermal relaxation time is very short compared to a period of oscillation). It is now clear that, when $|\omega| \gg |\eta s^2|$, we require for consistency that

$$q = \frac{\eta}{\kappa'} = \left|\frac{\eta s^2}{\omega}\right| \cdot \left|\frac{\omega}{\kappa' s^2}\right| \ll 1. \tag{4.12}$$

Thus the possibility that q might be greater than unity in (4.11) never arises. These observations do not mean that Busse's (1976) conclusion, that the direction of propagation of an instability depends upon the magnitudes of the diffusivities η and κ involved, is wrong. His result lies outside the scope of our eastward propagation theorem (which is for a compressible fluid and assumes that N^2 cannot be more than very marginally negative). The observations do, however, indicate that Busse's method cannot be extended to the present theory and is therefore of restricted validity.

Before proceeding, in the next section, to derive criteria for weakly diffusive instability, we define the isothermal and adiabatic scale heights

$$L_0 \equiv a_0^2/g, \quad L_* \equiv a_*^2/g \tag{4.13}$$

and the 'aspect length'

$$H \equiv D\frac{dy}{dz_i}. \tag{4.14}$$

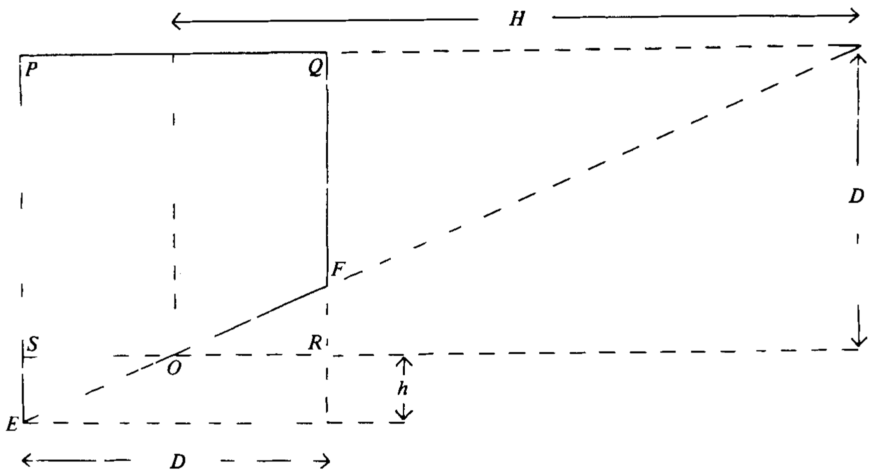

FIGURE 1. A cross-section of the geometrical configuration used in this paper; the x axis points out of the page. This shows that the aspect length H defined by (4.14) is the distance from the centre of the top to the point where its projection meets the projection of the (greatly exaggerated here) sloping bottom EF. O is the origin and the co-ordinates of the points P, Q, S, R in the y, z plane are, respectively, $(\pm\frac{1}{2}D, D)$ and $(\pm\frac{1}{2}D, 0)$; while the points E, F in which $z = z_l(y)$ meets the side walls are given by $\pm(\frac{1}{2}D, h)$.

The physical meaning of the aspect length is depicted by figure 1. Using (4.13) and (4.14) we define the instability parameter

$$\alpha \equiv \gamma\left(\frac{L_0-H}{L_*-H}\right) = \frac{\beta_0}{\beta_*}. \tag{4.15}$$

Thus α is the ratio of the isothermal to the adiabatic Rossby frequency and (since $\gamma > 1$) can be varied as follows to give any value except unity, as indicated in figure 1:

$$\left.\begin{array}{ll} H \leqslant L_0 & 0 \leqslant \alpha < 1; \\ L_0 \leqslant H < L_* & -\infty < \alpha \leqslant 0; \\ H > L_* & 1 < \alpha < \infty. \end{array}\right\} \tag{4.16}$$

The value $\alpha = 1$ would correspond to an incompressible fluid, for which the instabilities described in the following section cannot take place.

As we shall see, these instabilities are driven by the thermal diffusivity κ', they cannot take place if $\kappa' = 0$: for $\nu = \eta = 0$, this has been shown in § 3; for $\nu \neq 0$, $\eta \neq 0$ it can be verified by putting $\kappa' = 0$ in (4.5) and noting that this implies $\omega_{\text{Im}} < 0$.

5. Weak thermal diffusion: almost adiabatic modes

Let us assume that

$$\left|\frac{\Omega E^{\frac{1}{2}}}{\beta_*}\right| \sim \left|\frac{\eta s^2}{\beta_*}\right| \sim \left|\frac{\kappa s^2}{\beta_*}\right| \sim O(\epsilon) \tag{5.1}$$

and write the dispersion relation (2.61) in the form

$$\frac{\gamma N^2k^2}{s^2\beta_*^2}\left(\frac{\omega}{\beta_*}+\frac{i\eta s^2}{\beta_*}\right)+\frac{2\Omega kg}{a_0^2s^2\beta_*}\left(\frac{\omega}{\beta_*}+\frac{i\kappa s^2}{\beta_*}\right)\left(\frac{\omega}{\beta_*}+\frac{i\eta s^2}{\beta_*}\right)$$
$$+\frac{V^2k^2}{\beta_*^2}\left(\frac{\gamma\omega}{\beta_*}+\frac{i\kappa s^2}{\beta_*}\right)-\frac{2\Omega k}{s^2\beta_* D}\frac{dz_l}{dy}\left(\frac{\gamma\omega}{\beta_*}+\frac{i\kappa s^2}{\beta_*}\right)\left(\frac{\omega}{\beta_*}+\frac{i\eta s^2}{\beta_*}\right)$$
$$-\left(\frac{\omega}{\beta_*}+\frac{2\Omega iE^{\frac{1}{2}}}{\beta_*}\right)\left(\frac{\gamma\omega}{\beta_*}+\frac{i\kappa s^2}{\beta_*}\right)\left(\frac{\omega}{\beta_*}+\frac{i\eta s^2}{\beta_*}\right)=0. \tag{5.2}$$

Here, $\epsilon \ll 1$ is a small parameter in terms of which we shall expand the frequency ω:

$$\omega = \omega_0+\omega_1+O(\epsilon^2), \tag{5.3}$$

where $|\omega_1/\omega_0| = O(\epsilon)$. Let us now substitute (5.3) into (5.2) and keep all non-diffusive terms at order unity. At zeroth order in ϵ we obtain (3.4), which is conveniently rewritten in the form

$$\frac{\omega_0}{\beta_*}\left(\frac{\omega_0}{\beta_*}+1\right) = J_*; \tag{5.4}$$

for convenience, we repeat the definition (3.10):

$$J_* \equiv \beta_*^{-2}(V^2k^2+N^2k^2s^{-2}). \tag{5.5}$$

At first order in ϵ we obtain, on using (5.4):

$$-i\omega_1 s^{-2}\frac{\omega_0}{\beta_*}\left(\frac{2\omega_0}{\beta_*}+1\right) = \frac{N^2k^2}{s^2\beta_*^2}\eta+\frac{V^2k^2}{\beta_*^2}\kappa'-\frac{\omega_0}{\beta_*}(\alpha\kappa'+\eta)-\left(\frac{\omega_0}{\beta_*}\right)^2\left\{\eta+\kappa'+\frac{2\Omega E^{\frac{1}{2}}}{s^2}\right\}. \tag{5.6}$$

Since, by (5.4), ω_0/β_*+1 has the same sign as ω_0/β_*, the sign of $-i\omega_1$ is determined by the right-hand side of (5.6); using (5.4) again, the condition that the adiabatic waves (3.4) should amplify is therefore

$$(1-\alpha)\frac{\omega_0}{\beta_*} > \frac{N^2k^2}{\beta_*^2s^2}+q\frac{V^2k^2}{\beta_*^2}+P\left(\frac{\omega_0}{\beta_*}\right)^2. \tag{5.7}$$

The quantities κ', q and P are defined by (4.2) and (4.8). Let us now define the quantities ϕ:

$$\phi_{+,-} \equiv (1+4J_*)^{\frac{1}{2}} \pm 1. \tag{5.8}$$

The solutions (3.11) of the adiabatic dispersion relation (3.4) (or (5.4)) can then be written

$$\left(\frac{\omega_0}{\beta_*}\right)_{+,-} = \mp\tfrac{1}{2}\phi_{+,-}, \tag{5.9}$$

where, recalling our chosen terminology, the plus (minus) sign in (5.8) and (5.9) corresponds to the fast (slow) wave solution of the pair. Condition (5.7) for instability can now be written

$$\pm\frac{(\alpha-1)}{2} > \frac{\dfrac{N^2k^2}{\beta_*^2s^2}+q\dfrac{V^2k^2}{\beta_*^2}}{\phi_{+,-}}+\tfrac{1}{4}P\phi_{+,-}. \tag{5.10}$$

Since $\phi_{+,-} > 0$, the effect of viscosity is always stabilizing. (This remains true, of course, for the compressible analogue of Busse's model with interior viscosity and steep depth gradient (cf. appendix – then P is replaced by the Prandtl number ν/κ'.) To simplify presentation, therefore, we will henceforward set

$$\nu = 0, \tag{5.11}$$

and bear in mind that the growth rates calculated by (5.14) and (5.15) below would in practice be somewhat reduced by viscosity.

The growth rate, $-i\omega_1$, is now just $\kappa' s^2$ times the difference between the left- and right-hand sides of (5.10), with P set equal to zero. Let us denote the growth rates of the fast and slow waves, non-dimensionalized with respect to κ'/D^2, by

$$n_+ = \frac{D^2}{\kappa'}(-i\omega_1)_+, \quad n_- = \frac{D^2}{\kappa'}(-i\omega_1)_-. \tag{5.12}$$

Also, let us set

$$s^2 = k^2 + m^2 = r^2\pi^2/D^2, \quad r > 1; \tag{5.13}$$

$r > 1$ because π/D is the minimum value of m allowed by the boundary conditions (2.18) and $k \neq 0$ is assumed at (5.1). Then the growth rates n_+, n_- are given by

$$n_+ = \tfrac{1}{2}r^2\pi^2[\alpha - 1 - \tfrac{1}{2}a\phi_-] \tag{5.14}$$

and

$$n_- = \tfrac{1}{2}r^2\pi^2[1 - \alpha - \tfrac{1}{2}a\phi_+], \tag{5.15}$$

where

$$a(r^2) = \left(1 + q\frac{V^2r^2\pi^2}{N^2D^2}\right)\Big/\left(1 + \frac{V^2r^2\pi^2}{N^2D^2}\right). \tag{5.16}$$

We see that the fast wave can amplify only if $\alpha > 1$, while the slow wave can amplify only if $\alpha < 1$. Comparing (5.9) with the results of § 4 we see that this is just a manifestation of the fact that all amplifying waves must propagate eastward.

Once the parameters

$$\gamma, \ \alpha, \ q, \ \frac{D}{L_*}, \ \frac{N^2}{\Omega^2}, \ \frac{V^2}{\Omega^2 D^2}. \tag{5.17}$$

have been fixed, n is a function only of r^2 and can, in principle, be maximized by the use of ordinary differentiation with respect to r^2. However, this would involve the solution of an equation which is quintic in r^2; this cannot be performed algebraically. Thus, in the remainder of this section, we content ourselves with a numerical evaluation of n_+ and n_- for

$$\gamma = 1{\cdot}67, \quad \frac{D}{L_*} = 0{\cdot}1, \quad \frac{N^2}{\Omega^2} = \frac{V^2}{\Omega^2 D^2} = 10^{-5} \tag{5.18}$$

and various values of q; the results are drawn in figures 2 (*a*), (*b*) and (*c*) respectively for $\alpha = 0{\cdot}1$, $\alpha = -1{\cdot}67$ and $\alpha = 1{\cdot}67$. In each graph, the value of $0{\cdot}1n$ ($0{\cdot}1n_-$ in the first two cases and $0{\cdot}1n_+$ in the last) is plotted against values of r from $r = 1$ to $r = 6$. The r scale is the same in each diagram but $0{\cdot}1n$ is plotted for values from zero to $0{\cdot}1\bar{n}$ and the appropriate value of $\bar{n}$ is given in the figure caption. The three cases considered in figure 2 have been chosen to illustrate the three cases defined by (4.16), as follows.

(*a*) $H \leqslant L_0$ or $0 \leqslant \alpha < 1$. Only the slow wave can be amplified. From (5.15) it is clear that there can be no instability if either $q \geqslant 1$ or $V \neq 0$. Thus, although the instability is driven by thermal diffusion, the energy available for instability cannot be released without the help of the magnetic field.

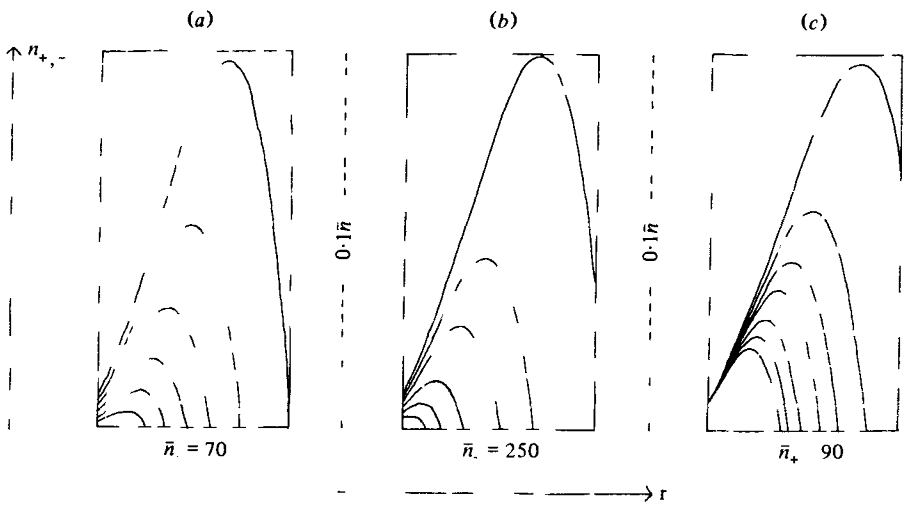

FIGURE 2. Graphs of the growth rate $0\cdot 1n$ for three values of the instability parameter α and for various values of the ratio q of the magnetic to the reduced thermal diffusivity, plotted against the wavenumber r; $n_{+,-}$ and r are defined by (5.13)–(5.15). A positive (negative) sign denotes a fast (slow) wave. In each case, the mapping rectangle is $\{1 \leqslant r \leqslant 6,\ 0 \leqslant 0\cdot 1n_{+,-} \leqslant 0\cdot 1\bar{n}\}$, where $\bar{n}$ is given, and the graph for the lowest value of q is uppermost, with graphs for lower values of q nested in sequence below it. (a) $\alpha = 0\cdot 1$, $q = 0\cdot 1$, $0\cdot 15$, $0\cdot 2$, $0\cdot 25$, $0\cdot 3$, $0\cdot 35$. (b) $\alpha = -1\cdot 67$, $q = 0\cdot 1$, $0\cdot 2$, $0\cdot 3$, $0\cdot 5$, $0\cdot 7$, $0\cdot 9$. (c) $\alpha = 1\cdot 67$, $q = 0\cdot 1$, $0\cdot 2$, $0\cdot 3$, $0\cdot 4$, $0\cdot 6$, $0\cdot 8$, $1\cdot 0$.

When $q = 0$, (5.15) predicts that the most rapidly growing mode has zero wavelength; thus some magnetic diffusion is needed to ensure that the critical wavenumber for maximum growth rate is finite. It is clear from figure 2(a) that, as q is increased from zero, this critical wavelength is increased from zero (r decreasing) but the corresponding maximum growth rate is at the same time reduced; and when η is so large that $q \gtrsim \frac{1}{2}$ the instability has been suppressed completely by magnetic diffusion.

(b) $L_0 \leqslant H < L_*$ *or* $-\infty < \alpha \leqslant 0$. Only the slow wave can be amplified. Instability can take place if $V = 0$ but some diffusion other than thermal diffusion (which is the destabilizing agent) is again needed to keep the critical wavenumber for maximum growth rate finite. In figure 2(b) $V \neq 0$ because η plays this role; however, since magnetic diffusion could be replaced by viscosity, the magnetic field is not essential for instability when α is negative.

Both the slow wave ($\alpha < 1$) instabilities we have just considered depend crucially upon the depth gradient since $\alpha = \gamma > 1$ when $dz_l/dy = 0$.

(c) $H > L_*$ *or* $1 < \alpha < \infty$. Only the fast wave can be amplified; apart from this, the only difference between the case $\alpha > 1$ and the case $\alpha < 0$ is that the depth gradient is *not* needed for the amplification of the fast wave. To illustrate this point in figure 2(c) we have chosen the value $\alpha = \gamma = 1\cdot 67$ corresponding to $dz_l/dy = 0$.

We emphasize, once again, that all three instabilities depend crucially upon compressibility: (5.10) could not be satisfied by taking $\alpha = 1$, which corresponds to an incompressible (Boussinesq) fluid. Thus the energy available for instability is attributable at least partly (and in cases (b) and (c) wholly) to the work done by compression of the fluid. Finally, we note that the third root of the cubic (5.2) is given to leading order by

$$\omega_3 = -is^2\kappa'\left(\frac{qN^2+V^2s^2}{N^2+V^2s^2}\right) \tag{5.19}$$

and thus always corresponds to decay.

6. Conclusion

We have studied linear wave motions in a rotating, compressible fluid which is capable of releasing elastic potential energy through instability. We have shown (§ 3) that compressibility allows a neutral Rossby wave to propagate even in the absence of a depth gradient; this cannot happen in a Boussinesq fluid. The Rossby waves may be amplified (§ 5) by a small amount of thermal diffusivity and all such instabilities must propagate eastward, independently of both the sign of the depth gradient and the magnitudes of all diffusivities involved.

The nature of the instabilities depends on whether the aspect length (see figure 1) is less than an isothermal scale height, greater than an adiabatic scale height or of intermediate size ($H < L_0$, $H > L_*$ or $L_0 < H < L_*$). When $H < L_0$, a condition which automatically requires the depth gradient to be non-zero and positive, the effects of 'curvature' (see the third paragraph of § 1) dominate over those of gravity and a magnetic field is needed to facilitate the release of elastic energy (§ 5). There is thus a good analogy with the magnetic field gradient instability, discussed by Acheson & Gibbons (1978), which occurs inside a certain critical radius in a cylindrical annulus of rotating, compressible fluid: this region is the one in which curvature effects dominate gravitational effects and these authors find that elastic potential energy can be released by a magnetic field which is perpendicular to both gravity and the rotation vector.

When $H > L_0$ and 'curvature' effects are dominated by gravity, a more direct release of elastic energy is possible (§ 5). A magnetic field ensures a finite maximum growth rate; but since viscosity could also do this, the magnetic field is not essential for instability. From an astrophysical point of view, this is certainly the most important case: whatever else may be necessary to apply our theory to the solar convection zone, where density varies rapidly with height, it is certain that we require $H > L_*$. Thus with reference to the question posed in the second paragraph of § 1, the present analysis effectively rules out (cf. Acheson 1979*a*) the possible importance of quasi-geostrophic magnetic instabilities in solar dynamics.

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Appendix. On the effect of viscosity

We have neglected viscous stresses in the interior of the fluid on the grounds that the wavelength of a disturbance is not too small, thus obviating the need to satisfy the condition that u_0 should vanish at the side-walls (which it will not do according to (2.60)). It is very simple to determine, however, what would happen if we ignored this boundary condition and, instead of Ekman suction, introduced a viscous stress at the same order as the Lorentz force in the momentum equation (2.19). Because of the local approximation, this stress would reduce (cf. Gibbons 1977; Acheson 1978*a*) to its incompressible counterpart $-\nu\nabla\wedge(\nabla\wedge\mathbf{u})$ and produce the following dispersion relation instead of (2.61):

$$-\frac{\gamma N^2k^2}{\gamma\omega+i\kappa s^2}+\frac{V^2k^2s^2}{\omega+i\eta s^2}-\frac{2\Omega k}{D}\frac{dz_l}{dy}-s^2(\omega+i\nu s^2)=\frac{-2\Omega kg}{a_0^2}\cdot\frac{\omega+i\kappa s^2}{\gamma\omega+i\kappa s^2}. \tag{A 1}$$

The only difference is that $2\Omega E^{\frac{1}{2}}$ is replaced by νs^2 in the fourth term of the left-hand side. Hence it is legitimate to neglect viscous stresses in the interior of the fluid provided only

$$\Omega E^{\frac{1}{2}} \gg \nu s^2 \quad \text{or} \quad \lambda \gg O(E^{\frac{1}{4}}D), \tag{A 2}$$

where λ is the disturbance wavelength and $E \ll 1$ (cf. Busse 1970).

It is reasonable to ask, however, what happens if the length scale in the y direction, Y say, is very much smaller than D. Then the relevant Ekman number, E_Y, is based on the length scale Y, not D; and if λ_Y is a disturbance wavelength in the y direction, viscosity in the interior of the fluid can be ignored only if

$$\lambda_Y \gg O(E_Y^{\frac{1}{4}}\, Y). \tag{A 3}$$

For $E \ll E_Y \ll 1$, $E_Y^{\frac{1}{4}}$ may well be close to unity, so that (A 3) will not be satisfied and viscosity will be more important in the interior of the fluid than in Ekman boundary layers (though properly, of course, we should include both). As recognized by Busse (1970), this ($Y \ll D$) situation is physically relevant because it allows the depth gradient to be of order unity, as it would be in a spherical shell, without destroying the two-dimensionality of the system. To see this, we have only to rescale the linearized equations: y is non-dimensionalized by Y ($\ll D$) instead of D. To ensure that w_0 still vanishes we must insist that $E_Y \ll 1$ as well; b_y and v must be rescaled (reduced) by a factor HD^{-1} to preserve Gauss's law and the continuity equation while t must be rescaled (lengthened) by a factor DH^{-1} to preserve geostrophic balance. Once all this has been done, the entire analysis of § 2 can be repeated and produces (A 1) if only s^2 is replaced, wherever it appears in (A 1), by m^2, to take account of the fact that $|k| \ll |m|$. Thus the two systems (gentle bottom slope and Ekman suction, order unity bottom slope and interior viscosity) are to a very large degree equivalent, at least under the local approximation, because we can switch from the first to the second just by replacing $2\Omega E^{\frac{1}{2}}$ and s^2 by νm^2 and m^2 respectively. We therefore expect that the results we have derived using the first system will be typical of the second system also (cf. Acheson 1978*b*).

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